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Cutting edge

Valuation adjustments

FVA for general instruments

A universal and efficient approach to numerical FVA calculation for portfolios of general instruments with multiple stochastic assets and funding sources

FVA for general instruments

Computing the funding valuation adjustment (FVA) is hard, as it requires the numerical solution of generally non-linear partial differential equations. In this paper, Alexander Antonov, Marco Bianchetti and Ion Mihai develop a universal and efficient approach to numerical FVA calculation for portfolios of general instruments with multiple stochastic assets and funding sources

One of the main lessons from the crisis has been that the price of financial instruments must include credit and funding risk components. The credit risk component accounts for the risk of default of the counterparties involved in the transaction, and leads to credit and debt valuation adjustments (CVA and DVA). The funding risk component accounts for the costs and benefits of the strategy adopted to borrow and lend the funds required or generated by the derivative, the hedge, and their possible collateralisations, and leads to funding valuation adjustment (FVA). The most important characteristic of the funding risk component is that it naturally involves multiple sources of funding, with corresponding multiple funding rates.

Bergman (1995) was the first author to extend the Black-Scholes framework to different lending and borrowing rates, Piterbarg (2010), in his seminal work, introduced repo and collateral rates, while Burgard & Kjaer (2013) added default hedges to the replication (see also Mercurio (2014) and Brigo *et al* (2014) for further details).

In this paper, we consider a pricing partial differential equation (PDE) with a general non-linear right-hand side recovering previous results as special cases. A numerical solution of such non-linear PDEs remains a complicated task even for simple instruments. We can cite two exceptions to this rule: the branching diffusion method of Henry-Labordere (2012) and the very recent forward Monte Carlo optimisation approach of Piterbarg (2015). However, none of them can be applied to the very general setup considered below.

We develop a practical numerical framework for computing the FVA of general financial instruments with path-dependent and callable features. Our approach is exact for all solvable special cases, ie, those where the collateral is a linear function of the instrument's price, and delivers a very accurate approximation for general instruments (barriers, Bermudans, etc). We also address the implementation workflow of the FVA calculation, explaining how our numerical framework permits parallel deal-by-deal computation, as opposed to portfolio-level optimisation. We also provide a concrete case study of FVA computation for a Bermudan swaption with partial collateralisation, showing that the quality of our numerical approximation is excellent.

Generalised pricing framework

The pricing framework can be summarised as follows (see Piterbarg 2010 and Antonov, Bianchetti & Mihai 2013). We postulate a risk-neutral measure $\mathbb{Q}$ and a general N-dimensional vector $\boldsymbol{r}$ of (short) funding rates following the Markovian dynamics $\mathrm{d}\boldsymbol{r}(t) = \mu(t, \boldsymbol{r})\,\mathrm{d}t + \sigma(t, \boldsymbol{r}) \cdot \mathrm{d}W(t)$, where $\mu(t, \boldsymbol{r})$ is a drift vector, $\sigma(t, \boldsymbol{r})$ is a volatility matrix, and $W(t)$ is a vector of independent Brownian motions under $\mathbb{Q}$ with associated expectation operator $\mathbb{E}[\cdots]$. We assume that our portfolio of derivatives can feature different types of exercises and payments that are functions of the rates $\boldsymbol{r}$. We denote by $V(t)$ and $C(t)$ the portfolio and collateral values at time t, respectively. The collateral amount $C(t)$ is a known, possibly non-linear, function of the portfolio value $V(t)$, as specified in the credit support annex (CSA) associated to the portfolio. The portfolio value $V(t)$ is a function of time and of the generalised model states, consisting of the rates $\boldsymbol{r}(t)$ and the underlying assets, such as equities, foreign exchange rates, etc. It satisfies the following generalised PDE:

$$\mathcal{L}(t)V(t) = \Omega(t, V(t)) \tag{1}$$

with possibly non-linear right-hand side (RHS). On the left-hand side we have denoted by $\mathcal{L}(t)$ the diffusion operator; when, for example, the vector of rates is composed of three rates $\boldsymbol{r} = \{r_{\mathrm{C}}, r_{\mathrm{R}}, r_{\mathrm{F}}\}$, for collateral, repo and unsecured funding, the operator has the form:

$$\mathcal{L}(t) = \partial_t + \sum_{\alpha} \mu_\alpha \partial_{r_\alpha} + \frac{1}{2} \sum_{\alpha,\beta} \sigma_\alpha \cdot \sigma_\beta \partial_{r_\alpha} \partial_{r_\beta}$$

where the sums run over $\alpha, \beta \in \{\mathrm{C}, \mathrm{R}, \mathrm{F}\}$. We stress that, since the funding strategy is defined at the portfolio level, the PDE (1) is intended for the whole portfolio and not just for an individual instrument. Thus, the notion of individual instrument is lost.

We can formally introduce the effective discount rate:

$$r_\Phi(t, V) = \frac{\Omega(t, V) - \Omega(t, 0)}{V} \tag{2}$$

which allows us to rewrite the pricing PDE (1) as:

$$\mathcal{L}(t)V(t) = r_\Phi(t, V(t))V(t) + \Omega(t, 0) \tag{3}$$

The RHS of the equation above is regularised for zero values of V: the extra term $\Omega(t, 0)$ has a sense of infinitesimal payments of $-\Omega(t, 0)\,\mathrm{d}t$.

An important example of the RHS, $\Omega(t, V) = r_{\mathrm{C}}C + r_{\mathrm{F}}(V - C)$, corresponds to a partial collateral case (Piterbarg (2010)). Going further, one can represent the RHS as a sum of different funding contributions with rate $r_\alpha(t)$ and amount $\Phi_\alpha(V(t))$:

$$\Omega(t, V(t)) = \sum_{\alpha} r_\alpha(t)\Phi_\alpha(V(t))$$

with the conservation $V(t) = \sum_\alpha \Phi_\alpha(V(t))$. One can find other examples of the non-linear RHS in Burgard & Kjaer (2013), Brigo *et al* (2014) and Antonov, Bianchetti & Mihai (2013).

By the Feynman-Kac theorem the solution of the PDE (1) can be written as an expectation under measure $\mathbb{Q}$ of future portfolio values discounted at the effective discount rate $r_\Phi(t, V)$.

Portfolio of general instruments and single-funding theory

In this section, we introduce a single-funding theory, as opposed to the multi-funding theory (3), and we define general financial instruments within this theory. The single-funding theory will be used as a calculation tool for our pricing approximation.

To set notation, we suppose that a Bermudan instrument k in the portfolio has payments of amounts $B_i^{(k)}$ on dates $t_i^{(k)}$, possibly depending on model rates r_α, and can be exercised into another instrument ℓ_k on dates $T_j^{(k)}$. For clarity, we assume that instrument ℓ_k itself has payments of amounts $B_i^{(\ell_k)}$ on dates $t_i^{(\ell_k)}$, but no exercises, ie, it is not callable.

We consider a single-funding model based on the discount rate $r(t)$, defined as a deterministic function of the multiple rates $\{r_\alpha(t)\}$. The portfolio value under this model, denoted by $v(t)$,[1] is additive, $v(t) = \sum_k v^{(k)}(t)$, and the value $v^{(k)}(t)$ of each individual kth instrument will satisfy the linear single-funding pricing PDE:

$$\mathcal{L}(t)v^{(k)}(t) = r(t)v^{(k)}(t) \tag{4}$$

between payment/exercise dates with the same evolution operator as in the PDE (3).

One can use delta functions to aggregate the cashflows and exercises into the RHS of the PDE:

$$\mathcal{L}(t)v^{(k)}(t) = r(t)v^{(k)}(t) - \sum_i \delta(t - t_i^{(k)})B_i^{(k)} - \sum_j \delta(t - T_j^{(k)})(v^{(\ell_k)}(t_+) - v^{(k)}(t_+))I_j^{(k)} \tag{5}$$

The first sum on the RHS corresponds to payments $B_i^{(k)}$ on dates $t_i^{(k)}$ and the second to exercises, with $I_j^{(k)}$ an exercise indicator, ie, we exercise into the instrument ℓ_k on date $T_j^{(k)}$ if $I_j^{(k)} = 1$. The optimal exercise indicator reads $I_j^{(k)} = \mathbb{I}_{v^{(\ell_k)}(t_+) > v^{(k)}(t_+)}$. The pricing equation for the ℓ_kth instrument is:

$$\mathcal{L}(t)v^{(\ell_k)}(t) = r(t)v^{(\ell_k)}(t) - \sum_i \delta(t - t_i^{(\ell_k)})B_i^{(\ell_k)} \tag{6}$$

Finally, we define the continuation and future values, which will play an important role in the FVA approximation described in the next section. The continuation value of an instrument at time t is the value of the instrument observed at t provided that it was not exercised before. We introduce the future value $v_t^{(k)}(u)$ as the value at time $u > t$ provided that the instrument was not exercised prior to time t. One can represent the future values via the continuation ones and global exercise indicators (see Antonov, Issakov & Mechkov (2015) and Antonov, Bianchetti & Mihai (2013) for more details).

It is important to note that PDEs involve pure continuation values while conditional expectations (eg, (7)) involve future values, which carry the information about possible exercises. We will expand on this remark in the next section.

[1] We denote with capital letters V, $V^{(k)}$ the portfolio and instrument values under the multi-funding setting, and by the corresponding lowercase letters v, $v^{(k)}$ their values under the single-funding model, respectively.

Funding valuation adjustment

In this section, we calculate the funding valuation adjustment (FVA) defined as the difference between the multi-funding value (1) and the single-funding value (4) of the portfolio, $\text{FVA}(t) := V(t) - v(t)$. We will use the single-funding model not only to calculate the base price, but also as our computational tool for evaluating the FVA.

Note that it is common market practice to define the FVA with respect to a perfectly collateralised price. In our framework, this choice corresponds to computing the single-funding value $v(t)$ using $r(t) = r_C(t)$ in (4). We will use this market convention in the next section.

The main technical result of this section is the FVA general formula for the portfolio:

$$\text{FVA}(t) \simeq -\mathbb{E}_t\biggl[\int_t^T \mathrm{d}u\,[\Omega(u, v_t(u)) - r(u)\,v_t(u)] \times \exp\biggl(-\int_t^u \mathrm{d}s\, r_\Phi(s, v_t(s))\biggr)\biggr] \tag{7}$$

Thus, to calculate the FVA, one should first obtain the portfolio future values $v_t(s)$ for the single-funding model and then aggregate them, taking into account the non-linear RHS of the multi-funding setup. The formula (7) is exact for a portfolio of non-callable instruments, provided that the multi-funding RHS is a linear function of the portfolio value. Other cases may be treated approximately with very high accuracy in two steps. First, we linearise the RHS of (3); second, we approximate the exercise behaviour in case of callable instruments.

Comparison with standard methods of FVA calculation. Unless the funding is simple (ie, leading to a linear RHS) the FVA calculation is complicated. Moreover, for instruments with exercises, 'market standard' ways to compute the FVA turn around the integral (7) but with different approximations of the discounting rate r_Φ: people replace it with the 'risk-free' rate r or use other ad hoc forms. Our solution with the non-linear rate r_Φ receives an extra order of accuracy, which we will discuss in the next paragraphs.

Linearisation. We apply the perturbation theory (Bogolyubov 2011) to the non-linear PDE introducing a small parameter related with the curve spreads. Suppose that the spreads can be written as $s_\alpha \equiv r_\alpha - r = \varepsilon\bar{s}_\alpha$ for small ε and for scaling coefficients $\bar{s}_\alpha \sim O(1)$. Then, the non-linear RHS can be written as the main linear part plus its perturbation:

$$\Omega(t, V) = rV + \varepsilon\Delta\Omega(t, V) \tag{8}$$

where $\Delta\Omega(t, V) = \sum_\alpha \bar{s}_\alpha(t)\Phi_\alpha(v) \sim O(1)$. Rewriting the main non-linear equation (3):

$$\mathcal{L}(t)V(t) = rV(t) + \varepsilon\Delta\Omega(t, V(t)) \tag{9}$$

we see that the leading term of the solution coincides with the single-funding price, ie, $V = v + O(\varepsilon)$, $\mathcal{L}v = rv$. According to the standard perturbation theory, we can obtain the first correction substituting the main order $v(t)$ into the RHS of (9). Instead, we will modify it in a slightly different way approximating its rate as $r_\Phi(t, V(t)) \simeq r_\Phi(t, v(t))$, ie:

$$\mathcal{L}(t)\tilde{V}(t) = r_\Phi(t, v(t))\tilde{V}(t) + \Omega(t, 0) \tag{10}$$

As we will see below, such modification will increase the approximation order, making the error in practical cases of the cubic order in the spreads, ie, $V = \tilde{V} + O(\varepsilon^3)$. This approximation efficiency is due to a special structure of the effective rates $r_\Phi(t, V)$. For most non-trivial CSAs (eg, one-sided) the rate is close to step-constant in V, such that its non-trivial derivatives (typically, δ-functions) are concentrated around zero values of V. This important result is proved in the appendix and confirmed numerically for challenging cases in the next section.

Exercise treatment. Now, our approximation $\tilde{V}(t)$ satisfies a linear PDE (10), but, as we will see below, we are not simply getting back to the single-funding theory, because the discount rate can still depend on instrument exercises.

In what follows, we will approximate the exercises under the multi-funding model by the single-funding one. For example, we exercise a Bermudan option if this decision is optimal from the point of view of the corresponding single-funding model. The reason for this is that the direct solution of the main PDE would require that each exercise option of each individual instrument in the portfolio should be calculated by a whole-portfolio optimisation.[2] This is a highly complex procedure, which, in practice, is intractable, and it is thus inevitable that we resort to an approximation. Furthermore, it is common market practice to exercise, or not exercise, individual callable instruments looking at the instrument itself and not at the portfolio they belong to.

Formally, given that $\tilde{V} - v \sim O(\varepsilon)$, the approximate exercise boundary will be shifted with respect to the exact one by $O(\varepsilon)$. As far as the exact exercise is supposed to be optimal, the price, as a function of exercise boundary, will have a maximum at the exact exercise boundary. This means the error in price will be of second order in ε.

Thus, the approximations above lead to the following PDE for the kth instrument:

$$\mathcal{L}(t)\tilde{V}^{(k)}(t) = r_\Phi(t, v(t))\tilde{V}^{(k)}(t) - \sum_i \delta(t - t_i^{(k)})B_i^{(k)} - \sum_j \delta(t - T_j^{(k)})(\tilde{V}^{(\ell_k)}(t_+) - \tilde{V}^{(k)}(t_+))I_j^{(k)} \quad (11)$$

where the exercise indicator $I_j^{(k)}$ is approximated using optimal consideration coming from the single-funding values, $I_j^{(k)} = \mathbb{I}_{v^{(\ell_k)}(t_+) > v^{(k)}(t_+)}$.

Let us emphasise that the effective rate $r_\Phi(t, v(t))$ in the PDE (11) is instrument-dependent. Imagine for simplicity that the portfolio contains only one instrument, k, which can be exercised into the ℓ_kth one. If we stay in instrument k (ie, we have not exercised prior to t), then the portfolio value $v(t)$ will be the value of instrument k: $v(t) = v^{(k)}(t)$. Otherwise, if we have already exercised into instrument ℓ_k, then $v(t) = v^{(\ell_k)}(t)$ and the rate $r_\Phi(t, v(t))$ depends now on $v^{(\ell_k)}(t)$. Omitting delta functions in the RHS, we obtain:

$$\left.\begin{aligned} \mathcal{L}(t)\tilde{V}^{(k)}(t) &= r_\Phi(t, v^{(k)}(t))\tilde{V}^{(k)}(t) \\ \mathcal{L}(t)\tilde{V}^{(\ell_k)}(t) &= r_\Phi(t, v^{(\ell_k)}(t))\tilde{V}^{(\ell_k)}(t) \end{aligned}\right\} \quad (12)$$

We see that the rate before exercise coincides with $r_\Phi(t, v^{(k)}(t))$, while the rate after the exercise is $r_\Phi(t, v^{(\ell_k)}(t))$. In both cases, these rates depend on the single-funding values v and are independent of the multi-funding values V. This situation corresponds to an option where the discount rate changes after the exercise. Note that such an option price can be calculated exactly provided that the exercise is automatic, ie, depends on the model states (eg, barrier, certain European, etc, but not Bermudan). The value $\tilde{V}^{(k)}(t)$ (12) is obtained given its single-funding future value $v_t^{(k)}$:

$$\tilde{V}^{(k)}(t) = v^{(k)}(t) - \mathbb{E}_t\left[\int_t^T \mathrm{d}u\,(r_\Phi(u, v_t^{(k)}(u)) - r(u))v_t^{(k)}(u) \times \exp\left(-\int_t^u \mathrm{d}s\, r_\Phi(s, v_t^{(k)}(s))\right)\right] \quad (13)$$

The proof can be found in Antonov, Bianchetti & Mihai (2013). We stress that the formula above uses the future value $v_t(u)$ in the effective rate, reflecting the fact that possible exercises change the rate. We warn against the use of the continuation value of the instrument $v^{(k)}(t)$ for the effective rate r_Φ calculation, ie, $r_\Phi(u, v^{(k)}(u))$ instead of $r_\Phi(u, v_t^{(k)}(u))$. As we have shown in Antonov, Bianchetti & Mihai (2013), such misuse can lead to significant errors.

For a general portfolio, containing multiple callable and non-callable instruments, we approximate $\mathrm{FVA}(t) = V(t) - v(t) \simeq \tilde{V}(t) - v(t)$ restoring (7).

Calculation scheme. The calculation workflow for the FVA at the origin has the following steps.

(1) Build the single-funding pricing model equipped with least-squares Monte Carlo and simulate the model rates $\{r_\alpha\}$, r and all payment indexes.

(2) Calculate future values $v_0^{(k)}(u)$ for all instruments in the portfolio on this model.[3] This can be done independently 'instrument-by-instrument' using the algorithmic exposure methods (Antonov, Issakov & Mechkov 2015) and enables thus the computation to be done in parallel.

(3) Sum up the instrument future values into the portfolio future values, $v_0(u) = \sum_k v_0^{(k)}(u)$, and calculate the FVA at origin using the expression (7) for $t = 0$ from the obtained future values $v_0(u)$.

Note also that the path dependence of the instruments can be included in the single-funding pricing mechanism. For this, it is sufficient to increase the regression space by extra path-dependent states.

For non-callable instruments, the FVA (7) is exact if the RHS of (3) is a linear function of V. Otherwise, it is an approximation. For a non-linear RHS, the exact price for a portfolio of non-callable instruments, as well as for an individual Bermudan instrument, can be obtained numerically by a small-timestep backward propagation (see the next section for more details).

Performance. The calculation scheme above shows that the biggest CPU usage falls on the future values calculation (step (2)). Indeed, for generic instruments, this calculation is based on the American Monte Carlo, a rather involved procedure. The aggregation of the future values in the integral (step (3)) is fast with respect to step (2). Indeed, for $t = 0$ there is no conditional expectation in the integral (7) and a simple Monte Carlo average is sufficient. Thus, the speed of

[2] This means that there is, numerically, a fundamental uncertainty in the FVA for exotic portfolios.

[3] As far as we are interested in the FVA at origin, we use the future values with zero subscript $v_0^{(k)}(u)$. This means all the exercises are taken into account.

our method is the same as for comparable 'market standard' methods (see the first paragraph of this section).

Numerical examples

In this section, we present the results of a few numerical experiments. We consider a 10Y Bermudan swaption that gives the owner the right to enter every year, starting with year 1, into a (co-terminal) swap, under which the owner receives a fixed rate annually and pays a Libor rate semi-annually on a 10,000 notional; the (first) swap is at-the-money.

To define the multi-funding setting we consider three rates: the collateral rate r_C, the (non-secured) funding rate r_F and an intermediate rate r_I. Their corresponding yield curves are defined by the continuous zero rates $R_\alpha(T)$ at the $T = 1\text{Y}$ and $T = 20\text{Y}$ pillars, with loglinear interpolation on discount factors $P_\alpha(0, T) = \mathrm{e}^{-R_\alpha(T)T}$ for $\alpha = \{C, F, I\}$.

The collateral rate is defined as follows: $R_C(1\text{Y}) = 1.5\%$ and $R_C(20\text{Y}) = 2.00\%$. The others will be defined below. We set the rate that underlies the base price to be the collateral rate, $r(t) = r_C(t)$, so that the FVA is computed with respect to a perfectly collateralised instrument. For the collateral rate, we postulate Hull-White one-factor dynamics with constant volatility of 1% and constant mean-reversion of 5%; for the other rates we use deterministic spreads with respect to the collateral rate.

Single-funding calculation. First, start with the single-funding (classical) calculation of the future values, the building blocks of the FVA. The calculations are performed with a backward Monte Carlo, where we average the results for several runs with different seeds; the future values under the single-funding model are computed using the algorithmic method from Antonov, Issakov & Mechkov (2015)

The mean of the future prices distribution (expected exposure), as well as the ± 1 standard deviation window, are shown in figure 1(a) for the swap (which we take to be the swap into which one can enter on the 1st exercise date) and figure 1(b) for the swaption.

Multi-curve exact calculation by backwards Monte Carlo. In different examples below we will need an exact multi-funding value to evaluate the accuracy of our FVA approximation. To compute this exact price corresponding to the multi-funding pricing PDE (3) we employ a fine-tenor backward propagation that takes into account the non-linear RHS and the optimal exercise condition. Indeed, formally, this PDE has solution:

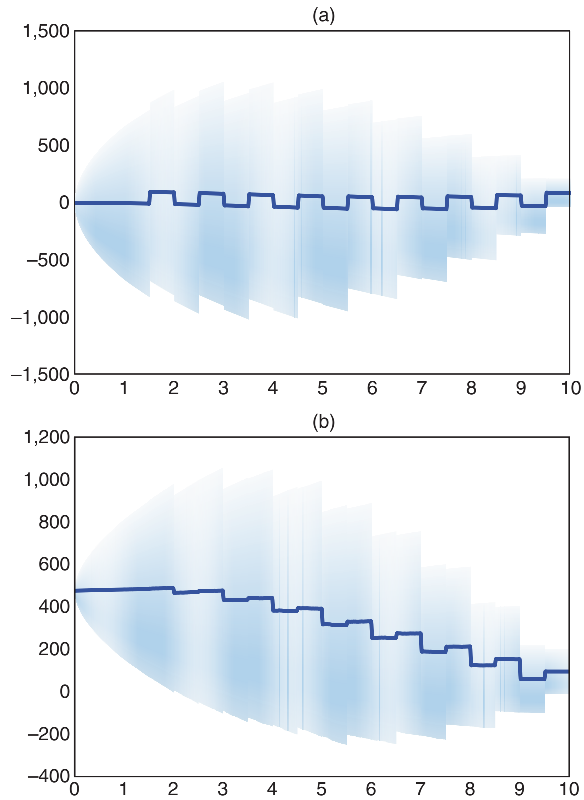
$$V(t) = \mathbb{E}[\mathrm{e}^{-\int_t^T r_\Phi(V(s))\,\mathrm{d}s} V(T) \mid \mathcal{F}_t]$$

Choosing a small spacing Δt we approximate:

$$\begin{aligned} V(t) &= \mathbb{E}[\mathrm{e}^{-\int_t^{t+\Delta t} r_\Phi(V(s))\,\mathrm{d}s} V(t+\Delta t) \mid \mathcal{F}_t] \\ &\simeq \mathbb{E}[\mathrm{e}^{-\int_t^{t+\Delta t} r(s)\,\mathrm{d}s} \mathrm{e}^{-\Delta t[r_\Phi(V(t+\Delta t)) - r(t+\Delta t)]} V(t+\Delta t) \mid \mathcal{F}_t] \end{aligned} \tag{14}$$

Thus, the exact price can be calculated with a backward discounted propagation by the single-funding model, taking into account the rate non-linearity at $t + \Delta t$. For this computation we use a 50 points-per-year time discretisation.

1 Expected exposure profiles: (a) swap and (b) swaption



The non-linearity of the effective rate makes the instrument pricing somewhat different from what is usual in the single-funding setting. For example, a payment at time T of an index I fixed at a previous time t cannot be simply added to the instrument leg L at time t, $L_{\text{new}}(t) \neq L(t) + P(t, T)I(t)$, where $P(t, T)$ is the zero-bond price. Instead, the non-linear effects force us to add the payments to legs when they are really paid, ie, $L_{\text{new}}(T) = L(T) + I(t)$. The subsequent backward propagation (14) should take into account the partial 'path-dependence' of the new value of the leg $L_{\text{new}}(T)$. Indeed, it will depend on the model states at time T as well as on the model states at time t via the index $I(t)$. This means that we should augment the space of regression variables: in our concrete case, for each observation date t in a period of the floating leg $[T_n, T_{n+1}]$, we add the Libor rate observed at T_n as an extra regression variable at t.

On each exercise date, we enter into the swap if the swaption exact continuation value is smaller than the swap exact value. Notice that we avoid a global optimisation as far as the portfolio contains a single callable instrument.

Below, we consider two numerical examples: a standard one and a challenging one. Both demonstrate the excellent approximation accuracy of the FVA (7).

Standard example. Similarly to numerical experiments with a simple two-rate asymmetric CSA from Antonov, Bianchetti & Mihai (2013) we consider the collateral $C(V) = (V - h)^+$ with a threshold $h = 350$ situated somewhere in the 'middle' of the Bermudan option (figure 1(b)). The funding rate defined by a flat spread of $s_F = R_F(t) - R_C(t) = 1.5\%$ at $t = 1\text{Y}$ and $t = 20\text{Y}$ is applied to the funding part

A. Collateral, intermediate rate and funding curves under the three scenarios						
Curve	Scenario 1		Scenario 2		Scenario 3	
	Rate (1Y)	Rate (20Y)	Rate (1Y)	Rate (20Y)	Rate (1Y)	Rate (20Y)
Int.	2.00%	2.25%	2.50%	2.50%	3.50%	3.00%
Funding	2.50%	2.50%	3.50%	3.00%	5.50%	4.00%

'Int.' denotes 'Intermediate'

B. Time-dependent threshold H									
Time (Y)	0	2	3	4	5	6	7	8	9
Threshold (H)	470	465	430	380	320	250	190	125	60

C. Swap results (notional amount 10,000)					
Scenario	Single-funding price	Multi-funding price	FVA	FVA approx	Diff.
1	0.00	8.32	8.32	8.32	0.00
2	0.00	16.24	16.24	16.23	−0.01
3	0.00	30.92	30.92	30.91	−0.01

D. Swaption results (notional amount 10,000)					
Scenario	Single-funding price	Multi-funding price	FVA	FVA approx	Diff.
1	470.05	465.49	−4.56	−4.56	0.00
2	470.05	460.95	−9.11	−9.08	0.03
3	470.05	451.98	−18.08	−17.99	0.09

E. Swaption prices for the exact RHS			
Scenario	Exact exercise	Approx. exercise	Diff.
1	465.49	465.50	0.00
2	460.95	460.97	0.02
3	451.98	452.05	0.07

$\min(V, h)$. Thus, the corresponding pricing equation (3) reads:

$$\mathcal{L}V = (V - h)^{+} r_{\mathrm{C}} + \min(V, h) r_{\mathrm{F}} = r_{\Phi}(V)V$$

We will concentrate on the swaption price whose single-funding and multi-funding exact values are, respectively, 470.05 and 446.10. This gives an exact FVA of −23.95. Our FVA approximation (7) is very close to the exact value, at −23.9. We also calculate one of possible 'standard' FVA values, where we replace the non-trivial rate r_{Φ} by the collateral rate r_{C} in the pricing integral (7); it occurs to be −27.6 leading to a difference of approximately 4 basis points. Relative to the FVA value itself, such mis-pricing is sizeable. Antonov, Bianchetti & Mihai (2013) provide other examples of the FVA approximation accuracy related with different approximations of the rate r_{Φ}.

Challenging example. In this example, we push the model settings to the most interesting conditions to 'stress-test' our approximation. We consider three different scenarios, defined as in table A, where we vary the steepness of the curves.

We define the non-linear RHS of the multi-funding pricing PDE (1) to be a piecewise linear function of the portfolio value V with two nodes, at 0 and at a threshold H; the slopes of this function on the three intervals $(-\infty, 0]$, $[0, H]$, $[H, +\infty)$ are the three rates r_{F}, r_{I}, r_{C}, respectively (see table A). This can also be regarded as defining the collateral function as $C(V) = (V - H)^{+}$ and, additionally, splitting the non-collateralised part $V - C$ into its positive and negative parts, with rates r_{I} and r_{F} respectively. The multi-funding pricing PDE therefore has the form:

$$\mathcal{L}V = r_{\mathrm{C}}(V - H)^{+} + r_{\mathrm{I}} \max(0, \min(H, V)) + r_{\mathrm{F}} \min(0, H, V) \quad (15)$$

The threshold H is chosen to be time-dependent, defined by the values given in table B, which are chosen to coincide with the mean future price of the swaption at each time horizon (expected exposure) in the single-funding model.

This example is very challenging: we fixed the rate non-linear points $\{0, H\}$ to the averages of the instruments, the swap and the swaption, and pushed the spreads up to 3%.

For each instrument we calculate the single-funding value, which is the solution of the single-funding pricing PDE (5), and the multi-funding, or exact, value, which corresponds to the multi-funding pricing PDE (15).

The difference between the multi-funding and single-funding values represents the 'true' FVA and is the benchmark against which we assess the FVA approximations below.

The FVA approximation is the one given by the main formula (7). The results are shown in table C for the swap (which we take to be the swap into which one can enter on the first exercise date) and table D for the Bermudan swaption. The FVA approximation values are listed in column 'FVA approx'. The true FVA values are listed in column 'FVA'; column 'Diff.' shows the difference between columns 'FVA approx' and 'FVA'.

Note that the swaption 'FVA approx' from table D contains two approximations: (1) the RHS linearisation; (2) exercise approximation by the single-funded exercise. To understand their relative contribution to the error, we compare the swaption prices with the exact (non-linear) RHS but with different exercises: one is exact (the same as in table D) while the second one is approximated. The results are given in table E.

We notice an excellent agreement of the approximate FVA with the benchmark true FVA, for both swaption and swap. We see that the main contributor to the swaption FVA error is the exercise approximation. For example, the scenario 3 total error of 9 basis points comes essentially from the exercise approximation (7bp). As mentioned in the main body of the article, such errors are inevitable.

To explain the excellent quality of our main approximation, we concentrate on two 'singularities' of the rate, $\{0, 500\}$. According to the appendix, the zero singularity gives a discrepancy of the third order in the spreads, while the 'fuzzy' singularity at H gives the second order of the accuracy. However, as also mentioned in the appendix, the swap average is far from H, thus, its second order error is attenuated with the small probability factor. The swaption error, gathered mainly on H, is of the second order. We observe it being two times bigger than the swap one, but still very small.

Conclusions

We presented a generalised pricing framework, where the replication portfolio is split between general, non-linear functions of the portfolio value, funded with different rates, based on Antonov, Bianchetti & Mihai (2013). We also proposed an implementation framework for the calculation of the FVA, which provides a practical and very accurate approximation for portfolios containing both vanilla and exotic instruments. Finally, we presented the numerical results for the FVA of a partially collateralised Bermudan swaption, showcasing the high accuracy of the approximation. All along, we maintained an understanding of theory tolerances, by keeping an eye on the amplitude of the approximation errors.

Appendix: approximation accuracy

In this appendix we prove that the difference between the exact and approximate solutions can be of the order $O(\varepsilon^3)$. Indeed, the difference $\tilde{J}(t) = \tilde{V}(t) - V(t)$ will satisfy:

$$\mathcal{L}(t)\tilde{J}(t) = r_\Phi(t, v(t))\tilde{J}(t) - (r_\Phi(t, V(t)) - r_\Phi(t, v(t)))V(t) \tag{A.1}$$

Noting that the instrument $\tilde{J}(t)$ has zero cashflows (they cancel out) we derive its value using the Feynmann-Kac formula similar to (7) and Antonov, Bianchetti & Mihai (2013):

$$\tilde{J}(t) = \mathbb{E}_t\bigg[\int_t^T \mathrm{d}u\,(r_\Phi(t, V_t(u)) - r_\Phi(t, v_t(u)))V_t(u) \times \exp\bigg(-\int_t^u \mathrm{d}s\, r_\Phi(s, v_t(s))\bigg)\bigg] \tag{A.2}$$

We stress that all the prices under the integral are future prices, not continuation values.

From (8) we deduce that $r_\Phi(t, V) = r + \varepsilon\Delta r_\Phi(t, V)$, where the 'perturbation' $\Delta r_\Phi(t, V) = V^{-1}(\Delta\Omega(t, V) - \Delta\Omega(t, 0)) \sim O(1)$. To evaluate the integral (A.2) order we concentrate on:

$$\begin{aligned} r_\Phi(t, V_t(u)) - r_\Phi(t, v_t(u)) &= \varepsilon(\Delta r_\Phi(t, V_t(u)) - \Delta r_\Phi(t, v_t(u))) \\ &\simeq \varepsilon\partial_v\Delta r_\Phi(t, v_t(u))\Delta v_t(u) \end{aligned}$$

where, as we know, the difference $\Delta v_t(u) \equiv V_t(u) - v_t(u) \sim O(\varepsilon)$. We suppose that the derivative $\partial_v\Delta r_\Phi(t, v)$ is concentrated around zero with respect to the product notional, which reflects a logic behind the collateral protection. For a typical case of the one-sided CSA with zero threshold, the effective rate perturbation will be simply $\Delta r_\Phi(t, v) = \bar{s}_\mathrm{F} 1_{V<0}$ for the funding spread $\bar{s}_\mathrm{F} = \varepsilon^{-1}(r_\mathrm{F} - r_\mathrm{C})$ and associated RHS $\Omega(t, v) = V^+ r_\mathrm{C} + V^- r_\mathrm{F}$. We treat the derivative $\partial_v\Delta r_\Phi(t, v)$ as proportional to $\delta(v)$. Thus, the order of the difference $\tilde{J}(t)$ looks like:

$$\begin{aligned} \tilde{J}(t) &\sim \mathbb{E}_t\bigg[\int_t^T \mathrm{d}u\,(r_\Phi(t, V_t(u)) - r_\Phi(t, v_t(u)))V_t(u)\bigg] \\ &\sim \varepsilon\mathbb{E}_t\bigg[\int_t^T \mathrm{d}u\,\partial_v\Delta r_\Phi(t, v_t(u))\Delta v_t(u)V_t(u)\bigg] \\ &\sim \varepsilon\mathbb{E}_t\bigg[\int_t^T \mathrm{d}u\,\delta(v_t(u))\Delta v_t(u)V_t(u)\bigg] \\ &\sim \varepsilon\int_t^T \mathrm{d}u\, P_{v_t(u)}(0)\mathbb{E}_t[(\Delta v_t(u))^2 \mid v_t(u) = 0] \sim O(\varepsilon^3) \end{aligned} \tag{A.3}$$

where $P_{v_t(u)}(0) = \mathbb{E}[\delta(v_t(u))]$ is a probability density of $v_t(u)$ taking zero value. Below we have used the fact that:

$$\begin{aligned} &\mathbb{E}[\delta(v_t(u))\Delta v_t(u)V_t(u)] \\ &\quad = \mathbb{E}[\delta(v_t(u))]\mathbb{E}[\Delta v_t(u)V_t(u) \mid v_t(u) = 0] \\ &\quad = P_{v_t(u)}(0)\mathbb{E}[(\Delta v_t(u))^2 \mid v_t(u) = 0] \end{aligned}$$

Thus, if the effective rate has a step-constant form with the singularity at zero, the exact solution is approximated with $\tilde{V}(t)$ with a very high accuracy, $V = \tilde{V} + O(\varepsilon^3)$. Otherwise, if the rate singularity is far from zero, the accuracy will be of second order proportional to a probability density of the future values $v_t(u)$ being on the rate singularity (as we see from the integral (A.3) when we replace the zero rate singularity by that in hand).

If the rate derivative is concentrated over several (singular) points, the overall accuracy is a sum over the integrals (A.3). ■

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